

# Measurable centres in convolution semigroups

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## Abstract

In a convolution semigroup over a locally compact group, measurability of the translation by a fixed element implies continuity. In other words, the measurable centre coincides with the topological centre.

## 1 Overview

For a topological group  $G$ , let  $rG$  denote  $G$  with its right uniformity, and  $\text{LUC}(G)$  the space of bounded uniformly continuous functions on  $rG$ . The norm dual  $\text{LUC}(G)^*$  of  $\text{LUC}(G)$  with convolution  $\star$  is a Banach algebra of some importance in harmonic analysis. A useful tool for investigating the structure of  $\text{LUC}(G)^*$  is its *topological centre*

$$\Lambda(\text{LUC}(G)^*) = \{\mathfrak{m} \in \text{LUC}(G)^* \mid \text{the mapping } \mathfrak{n} \mapsto \mathfrak{m} \star \mathfrak{n} \text{ is weak}^* \text{ continuous on } \text{LUC}(G)^*\}.$$

The space  $\mathbf{M}_t(G)$  of (finite signed) Radon measures on  $G$  naturally embeds in  $\text{LUC}(G)^*$ : A measure  $\mu \in \mathbf{M}_t(G)$  maps to the functional  $f \mapsto \int f \, d\mu$ ,  $f \in \text{LUC}(G)$ . By the theorem of Lau [12],  $\Lambda(\text{LUC}(G)^*) = \mathbf{M}_t(G)$  for every locally compact group  $G$ .

In this paper I prove a stronger version of Lau's result, in which weak\* continuity in the definition of  $\Lambda(\text{LUC}(G)^*)$  is replaced by measurability. I also prove similar characterizations for such generalized (measurable) centres of subsemigroups of  $\text{LUC}(G)^*$ . In particular, the result applies to the semigroup  $\beta G$  for any discrete group  $G$ , thus extending a recent result of Glasner [10, Th.2.1].

## 2 Preliminaries

All topological spaces and groups considered in this paper are assumed to be Hausdorff, and all linear spaces to be over the field  $\mathbb{R}$  of reals. Functions (including linear functionals) are real-valued. It is a simple exercise to extend the results that follow to linear spaces over the complex field and complex-valued functions.

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A pseudometric  $\Delta$  on a group  $G$  is *right-invariant* iff  $\Delta(x, y) = \Delta(xz, yz)$  for all  $x, y$  and  $z$  in  $G$ . The *right uniformity* on a topological group  $G$  is induced by the set of all right-invariant continuous pseudometrics, denoted by  $\text{RP}(G)$ . A bounded pseudometric  $\Delta$  on  $G$  is uniformly continuous in  $rG$  if and only if there exists  $\Delta' \in \text{RP}(G)$  such that  $\Delta \leq \Delta'$ .

When  $\Delta$  is a pseudometric on  $G$ , write

$$\text{BLip}_b(\Delta) = \{f: G \rightarrow \mathbb{R} \mid -1 \leq f(x) \leq 1 \text{ and } |f(x) - f(y)| \leq \Delta(x, y) \text{ for all } x, y \in G\}.$$

Then  $\text{BLip}_b(\Delta)$  is a compact subset of the product space  $\mathbb{R}^G$ ; in the sequel  $\text{BLip}_b(\Delta)$  is always considered with this compact topology.

Let  $G$  be a group,  $f$  a real-valued function on  $G$  and  $x \in G$ . Define  $\rho^x(f)$  (the *right translation* of  $f$  by  $x$ ) to be the function  $z \mapsto f(zx)$ ,  $z \in G$ . The set  $\text{orb}(f) := \{\rho^x(f) \mid x \in G\}$  is the (*right*) *orbit* of  $f$ . The closure of  $\text{orb}(f)$  in the product space  $\mathbb{R}^G$  is denoted  $\overline{\text{orb}}(f)$ . For every  $f \in \text{LUC}(G)$  the set  $\text{orb}(f)$  is norm bounded and uniformly equicontinuous on  $rG$ , and thus  $\overline{\text{orb}}(f)$  is a  $G$ -pointwise compact subset of  $\text{LUC}(G)$ .

**Fact 2.1 (Cor. 15 in [15])** *Let  $G$  be a locally compact group that is not compact. For every  $\Delta \in \text{RP}(G)$  there is  $f \in \text{LUC}(G)$  such that  $\text{BLip}_b(\Delta) \subseteq \overline{\text{orb}}(f)$ .*

**Fact 2.2 ([1] and [5])** *Let  $G$  be a locally compact group and  $\mathbf{m} \in \text{LUC}(G)^*$ . If the restriction of  $\mathbf{m}$  to the set  $\text{BLip}_b(\Delta)$  is continuous for every  $\Delta \in \text{RP}(G)$  then  $\mathbf{m} \in \mathbf{M}_t(G)$ .*

By combining the first two facts we obtain a characterization of finite Radon measures on locally compact groups: A functional  $\mathbf{m} \in \text{LUC}(G)^*$  is in  $\mathbf{M}_t(G)$  if and only if for every  $f \in \text{LUC}(G)$  the restriction of  $\mathbf{m}$  to  $\overline{\text{orb}}(f)$  is  $G$ -pointwise continuous.

Let  $X$  be a (Hausdorff as always) topological space and  $A \subseteq X$ . Say that  $A$  is a *CBP set* iff for every continuous mapping  $\varphi: K \rightarrow X$  from a compact space  $K$  the set  $\varphi^{-1}(A)$  has the Baire property in  $K$ . Say that a real-valued function  $f$  on  $X$  is *CBP measurable* iff  $f^{-1}(U)$  is a CBP set in  $X$  for every open subset  $U$  of  $\mathbb{R}$ .

When  $X$  is compact, CBP subsets of  $X$  are exactly the universally Baire-property sets in the terminology of Fremlin [9]. Evidently the CBP subsets of  $X$  form a  $\sigma$ -algebra, every Borel set is CBP, and every Borel measurable mapping is CBP measurable. If a mapping  $f: X \rightarrow Y$  is CBP measurable then so is its restriction to any subspace of  $X$ .

Denote by  $\mathcal{P}I$  the set of all subsets of a set  $I$ , and identify  $\mathcal{P}I$  with the compact set  $2^I$ .

**Fact 2.3 (Lemma 2.1 in [3])** *Let  $I$  be an infinite set, and let  $\mu: \mathcal{P}I \rightarrow \mathbb{R}$  be finitely additive. If  $\mu$  is CBP measurable then it is a measure on  $\mathcal{P}I$  and  $\mu(A) = \sum_{i \in A} \mu(\{i\})$  for every  $A \subseteq I$ .*

The next theorem and its proof are due to Fremlin. A slightly different version appears in [9, 1E].

**Theorem 2.4** *Let  $K_0$  be a compact space and  $\varphi_0: K_0 \rightarrow X$  a continuous surjective mapping onto a compact space  $X$ . Let  $A \subseteq X$  be such that  $\varphi_0^{-1}(A)$  is a CBP set in  $K_0$ . Then  $A$  is a CBP set in  $X$ .*

**Proof.** Take any continuous mapping  $\varphi_1: K_1 \rightarrow X$  from a compact space  $K_1$ . For  $i = 0, 1$  let  $\pi_i: K_0 \times K_1 \rightarrow K_i$  be the canonical projections. Then

$$K := \{(x_0, x_1) \in K_0 \times K_1 \mid \varphi_0(x_0) = \varphi_1(x_1)\}$$

is a compact subset of  $K_0 \times K_1$ , and  $\pi_1(K) = K_1$  because  $\varphi_0(K_0) = X$ . By 4A2Gi in [7] there is a closed set  $K' \subseteq K$  such that the restriction of  $\pi_1$  to  $K'$  is irreducible and  $\pi_1(K') = K_1$ .

By [17, L.2], [19, 25.2.3], if a set has the Baire property in a compact space, then so does its image under any irreducible continuous surjection. Since  $\varphi_0^{-1}(A)$  is a CBP set, the set  $K' \cap \pi_0^{-1}(\varphi_0^{-1}(A))$  has the Baire property in  $K'$ , and the set

$$\pi_1(K' \cap \pi_0^{-1}(\varphi_0^{-1}(A))) = \varphi_1^{-1}(A)$$

has the Baire property in  $K_1$ . □

The following theorem is an essential step in the proof of the main result in the next section. As before,  $\text{BLip}_b(\Delta)$  is considered with the  $G$ -pointwise topology.

**Theorem 2.5** *Let  $G$  be a locally compact group and  $\mathfrak{m} \in \text{LUC}(G)^*$ , and assume that for every  $\Delta \in \text{RP}(G)$  the restriction of  $\mathfrak{m}$  to  $\text{BLip}_b(\Delta)$  is CBP measurable. Then for every  $\Delta \in \text{RP}(G)$  the restriction of  $\mathfrak{m}$  to  $\text{BLip}_b(\Delta)$  is continuous.*

For metrizable locally compact groups, Theorem 2.5 is a direct consequence of Theorem 2 and Lemma 4.1 in [3]. Here I prove the general case, after several auxiliary lemmas.

A *partition of unity* on a set  $X$  is a mapping  $p: X \rightarrow \ell_\infty(I)$  where  $I$  is a non-empty index set,  $0 \leq p(x)(i) \leq 1$  for all  $x \in X$  and  $i \in I$ , and  $\sum_{i \in I} p(x)(i) = 1$  for every  $x \in X$ . Write  $p_i(x) := p(x)(i)$ ; thus each  $p_i$  is a function on  $X$  with values in the interval  $[0, 1]$ . Note that the range of  $p$  is included in  $\ell_1(I) \subseteq \ell_\infty(I)$ . Denote the  $\ell_1(I)$  norm by  $\|\cdot\|_1$  and the  $\ell_\infty(I)$  norm by  $\|\cdot\|_\infty$ .

When  $\Delta$  is a pseudometric on  $X$ , say that the partition of unity  $p: X \rightarrow \ell_\infty(I)$  is *subordinated to  $\Delta$*  iff for every  $i \in I$  we have  $\Delta(x, y) \leq 1$  whenever  $x, y \in G$ ,  $p_i(x) > 0$ ,  $p_i(y) > 0$ . When  $G$  is a topological group, say that the uniform space  $rG$  has the  $(\ell_1)$  *property* iff for every  $\Delta \in \text{RP}(G)$  there exists a partition of unity  $p$  on  $G$  that is subordinated to  $\Delta$  and uniformly continuous from  $rG$  to  $\ell_1(I)$  with the  $\|\cdot\|_1$  norm.

**Lemma 2.6** *For every locally compact group  $G$  the uniform space  $rG$  has the  $(\ell_1)$  property.*

For metrizable locally compact groups this is Lemma 4.1 in [3]. Essentially the same proof works for the general case, and I do not repeat it here.

**Lemma 2.7** *Let  $G$  be a topological group and let  $\mathfrak{m} \in \text{LUC}(G)^*$  be such that for every  $\Delta \in \text{RP}(G)$  the restriction of  $\mathfrak{m}$  to  $\text{BLip}_b(\Delta)$  is CBP measurable. Let  $I$  be a non-empty index set and  $\varphi: rG \rightarrow \ell_1(I)$  a uniformly continuous mapping from  $rG$  to  $\ell_1(I)$  with the  $\|\cdot\|_1$  norm, and such that  $\|\varphi(x)\|_1 \leq 1$  for every  $x \in G$ . Then*

$$\sum_{i \in I} |\mathfrak{m}(\varphi_i)| < \infty \quad \text{and} \quad \sum_{i \in A} \mathfrak{m}(\varphi_i) = \mathfrak{m} \left( \sum_{i \in A} \varphi_i \right) \quad \text{for } A \subseteq I$$

where  $\varphi_i(x) := \varphi(x)(i)$  for  $i \in I$ ,  $x \in X$ .

**Proof.** Since  $\varphi$  is uniformly continuous in the  $\|\cdot\|_1$  norm, there is  $\Delta \in \text{RP}(G)$  such that  $\|\varphi(x) - \varphi(y)\|_1 \leq \Delta(x, y)$  for all  $x, y \in G$ . The expression

$$\psi(A) := \sum_{i \in A} \varphi_i(x), \quad A \subseteq I,$$

defines a continuous finitely additive mapping  $\psi: \mathcal{P}I \rightarrow \text{BLip}_b(\Delta)$ . Hence the function  $\mathbf{m} \circ \psi$  is finitely additive and CBP measurable on  $\mathcal{P}I$ . Apply Fact 2.3.  $\square$

**Proof of Theorem 2.5.** Take any  $\Delta \in \text{RP}(G)$  and any net  $\{f_\gamma\}_\gamma$  of functions  $f_\gamma \in \text{BLip}_b(\Delta)$  such that  $\lim_\gamma f_\gamma(x) = 0$  for all  $x \in G$ . Fix an arbitrary  $\varepsilon > 0$ . By Lemma 2.6 there is a partition of unity  $p$  on  $G$  that is subordinated to  $\Delta/\varepsilon$  and uniformly continuous from  $rG$  to  $\ell_1(I)$  with the  $\|\cdot\|_1$  norm.

For each  $i \in I$  choose a point  $x_i \in G$  such that  $\Delta(x, x_i) \leq \varepsilon$  whenever  $p_i(x) > 0$ . Then

$$\left| f_\gamma(x) - \sum_{i \in I} f_\gamma(x_i) \cdot p_i(x) \right| \leq \sum_{i \in I} p_i(x) \cdot |f_\gamma(x) - f_\gamma(x_i)| \leq \sum_{i \in I} p_i(x) \cdot \Delta(x, x_i) \leq \varepsilon$$

for all  $\gamma$  and all  $x \in G$ .

For a fixed  $\gamma$ , define  $\varphi: G \rightarrow \ell_1(I)$  by  $\varphi(x)(i) := f_\gamma(x_i) \cdot p_i(x)$ ,  $x \in G$ ,  $i \in I$ , and apply Lemma 2.7 to get

$$\sum_{i \in I} f_\gamma(x_i) \mathbf{m}(p_i) = \mathbf{m} \left( \sum_{i \in I} f_\gamma(x_i) p_i \right).$$

By Lemma 2.7 there is a finite set  $D \subseteq I$  such that  $\sum_{i \in I \setminus D} |\mathbf{m}(p_i)| < \varepsilon$ . For almost all  $\gamma$  we have  $|f_\gamma(x_i)| < \varepsilon$  when  $i \in D$ , and

$$\begin{aligned} |\mathbf{m}(f_\gamma)| &\leq \left| \mathbf{m} \left( f_\gamma - \sum_{i \in I} f_\gamma(x_i) \cdot p_i \right) \right| + \left| \sum_{i \in I} f_\gamma(x_i) \mathbf{m}(p_i) \right| \\ &\leq \|\mathbf{m}\| \varepsilon + \sum_{i \in D} |f_\gamma(x_i)| \cdot |\mathbf{m}(p_i)| + \sum_{i \in I \setminus D} |f_\gamma(x_i)| \cdot |\mathbf{m}(p_i)| \\ &\leq \|\mathbf{m}\| \varepsilon + 2\|\mathbf{m}\| \varepsilon + \varepsilon = (3\|\mathbf{m}\| + 1) \varepsilon. \end{aligned}$$

As this holds for every  $\varepsilon > 0$ ,  $\mathbf{m}$  is continuous on  $\text{BLip}_b(\Delta)$ .  $\square$

### 3 Generalized centres

For any topological group  $G$ , the convolution in  $\text{LUC}(G)^*$  may be written as

$$\mathbf{m} \star \mathbf{n}(f) = \mathbf{m}(\backslash_x \mathbf{n}(\backslash_y f(xy)))$$

for  $\mathbf{m}, \mathbf{n} \in \text{LUC}(G)^*$  and  $f \in \text{LUC}(G)$ . Here  $\backslash_x f(\dots)$  means “ $f(\dots)$  as a function of  $x$ ”. This formula applies not only in  $\text{LUC}(G)^*$  for a topological group  $G$  but also in analogous spaces over more general *semiuniform semigroups* [14].

$\text{LUC}(G)^*$  with convolution is a Banach algebra. Here we mostly treat it as a semigroup with the  $\star$  operation. The group  $G$  naturally embeds in  $\text{LUC}(G)^*$ : An element  $x \in G$  maps to the functional  $f \mapsto f(x)$ ,  $f \in \text{LUC}(G)$ . The embedding is a homeomorphism of  $G$  onto its image in  $\text{LUC}(G)^*$  with the weak\* topology. The embedding also preserves the algebraic structure, so that  $G$  may be identified with a subgroup of  $\text{LUC}(G)^*$ .

The weak\* closure of  $G$  in  $\text{LUC}(G)^*$ , denoted here  $G^{\text{LUC}}$ , is a weak\* compact subsemigroup of  $\text{LUC}(G)^*$ . It is known as the *canonical  $\mathcal{LC}$ -compactification* [2], *universal enveloping semigroup* [4],  *$\mathcal{LUC}$ -compactification* [13], or *greatest ambit* [16] of  $G$ ; or, in the language of uniform spaces, a *uniform (or Samuel) compactification* of  $rG$ . When  $G$  is discrete,  $G^{\text{LUC}}$  is its Čech–Stone compactification  $\beta G$ . When  $G$  is locally compact,  $G^{\text{LUC}} \cap \mathbf{M}_t(G) = G$ .

For any topological group  $G$  and any  $S \subseteq \text{LUC}(G)^*$  define

$$\Lambda(S) := \{\mathbf{m} \in S \mid \forall f \in \text{LUC}(G) \text{ the function } \mathbf{n} \mapsto \mathbf{m} \star \mathbf{n}(f) \\ \text{is weak* continuous on } S\}$$

$$\Lambda^{\text{CBP}}(S) := \{\mathbf{m} \in S \mid \forall f \in \text{LUC}(G) \text{ the function } \mathbf{n} \mapsto \mathbf{m} \star \mathbf{n}(f) \\ \text{is weak* CBP measurable on } S\}$$

(the *topological centre* and the *weak\* CBP measurable centre* of  $S$ ).

It is well known and easy to prove that  $S \cap \mathbf{M}_t(G) \subseteq \Lambda(S) \subseteq \Lambda^{\text{CBP}}(S) \subseteq S$ . If  $G$  is compact then  $\text{LUC}(G)^* = \mathbf{M}_t(G)$  and therefore  $\Lambda(S) = \Lambda^{\text{CBP}}(S) = S = S \cap \mathbf{M}_t(G)$  for every  $S \subseteq \text{LUC}(G)^*$ .

Now we come to the main result of this paper. The proof strategy is the same as in section 5 of [15].

**Theorem 3.1** *Let  $G$  be a locally compact group and  $G^{\text{LUC}} \subseteq S \subseteq \text{LUC}(G)^*$ . Then*

$$\Lambda(S) = \Lambda^{\text{CBP}}(S) = S \cap \mathbf{M}_t(G).$$

**Proof.** In view of the preceding discussion, it is enough to prove that  $\Lambda^{\text{CBP}}(S) \subseteq S \cap \mathbf{M}_t(G)$  when  $G$  is not compact.

For  $f \in \text{LUC}(G)$  define the mapping  $\varphi_f: \text{LUC}(G)^* \rightarrow \text{LUC}(G)$  by

$$\varphi_f(\mathbf{n}) := \backslash_x \mathbf{n}(\backslash_y f(xy)), \quad \mathbf{n} \in \text{LUC}(G)^*.$$

Then for every  $\mathbf{m} \in \text{LUC}(G)^*$  the mapping  $\mathbf{n} \mapsto \mathbf{m} \star \mathbf{n}(f)$  from  $\text{LUC}(G)^*$  to  $\mathbb{R}$  is the composition  $\mathbf{m} \circ \varphi_f$ . By [15, L.19],  $\varphi_f$  is continuous from  $G^{\text{LUC}}$  to  $\text{LUC}(G)$  with the  $G$ -pointwise topology, and  $\varphi_f(G^{\text{LUC}}) = \overline{\text{orb}}(f)$ .

$$\begin{array}{ccc} G^{\text{LUC}} & \xrightarrow{\varphi_f} & \overline{\text{orb}}(f) \\ & \searrow \mathbf{n} \mapsto \mathbf{m} \star \mathbf{n}(f) & \downarrow \mathbf{m} \\ & & \mathbb{R} \end{array}$$

Now assume that  $\mathbf{m} \in \Lambda^{\text{CBP}}(S)$ , which means that for every  $f \in \text{LUC}(G)$  the mapping  $\mathbf{m} \circ \varphi_f$  is CBP measurable on  $S$ , and therefore also on  $G^{\text{LUC}} \subseteq S$ . By Theorem 2.4,  $\mathbf{m}$  is CBP measurable

on  $\overline{\text{orb}}(f)$ . By Fact 2.1,  $\mathbf{m}$  is CBP measurable on  $\text{BLip}_b(\Delta)$  for every right-invariant continuous pseudometric  $\Delta$  on  $G$ , and therefore also continuous on  $\text{BLip}_b(\Delta)$  by Theorem 2.5. Hence  $\mathbf{m} \in \mathbf{M}_t(G)$  by Fact 2.2.  $\square$

By choosing  $S = \text{LUC}(G)^*$  and  $S = G^{\text{LUC}}$  we obtain two corollaries. The first one is the promised strengthening of Lau's theorem [12].

**Corollary 3.2**  $\Lambda(\text{LUC}(G)^*) = \Lambda^{\text{CBP}}(\text{LUC}(G)^*) = \mathbf{M}_t(G)$  for every locally compact group  $G$ .

The second corollary is a common generalization of the theorems of Lau and Pym [13] and Glasner [10].

**Corollary 3.3**  $\Lambda(G^{\text{LUC}}) = \Lambda^{\text{CBP}}(G^{\text{LUC}}) = G$  for every locally compact group  $G$ .

Note that Theorem 3.1 applies also to many other sets between  $G^{\text{LUC}}$  and  $\text{LUC}(G)^*$  — for example, the set of positive elements in  $\text{LUC}(G)^*$ , or the set of means on  $\text{LUC}(G)$ .

## 4 Variations and open problems

One may ask to what extent Theorem 3.1 and its corollaries depend on the group  $G$  being locally compact. The results in [6] and [15] suggest that the space  $\mathbf{M}_u(rG)$  of *uniform measures* should take the place of  $\mathbf{M}_t(G)$  in describing the centres in convolution semigroups as we move beyond locally compact groups ( $\mathbf{M}_u(rG)$  and  $\mathbf{M}_t(G)$  coincide when  $G$  is locally compact). This leads to the question whether  $\Lambda^{\text{CBP}}(\text{LUC}(G)^*) = \mathbf{M}_u(rG)$  for every topological group  $G$ , or at least for some interesting class of non-locally-compact groups.

With the same approach as in the proof of Theorem 3.1, we get that  $\Lambda^{\text{CBP}}(S) = S \cap \mathbf{M}_u(rG)$  for  $G^{\text{LUC}} \subseteq S \subseteq \text{LUC}(G)^*$  whenever  $G$  is an ambitable topological group [15] for which  $rG$  has the  $(\ell_1)$  property. However, infinite-dimensional normed spaces do not have the  $(\ell_1)$  property by the theorem of Zahradník [20].

One may also try to weaken the measurability condition in the definition of  $\Lambda^{\text{CBP}}(S)$ . In one direction, Schachermayer's example [18] marks a limit of such generalizations: For the additive group  $c_0$  and the metric  $\Delta$  of the sup norm on  $c_0$ , there is a bounded linear functional  $\mathbf{m}$  on  $\text{LUC}(c_0)$  whose restriction to  $\text{BLip}_b(\Delta)$  is Baire-property measurable and yet  $\mathbf{m}$  is not in  $\mathbf{M}_t(c_0)$ .

In another direction, for  $S \subseteq \text{LUC}(G)^*$  define

$$\Lambda^{\text{URM}}(S) = \{\mathbf{m} \in S \mid \forall f \in \text{LUC}(G) \text{ the function } \mathbf{n} \mapsto \mathbf{m} \star \mathbf{n}(f) \text{ is weak}^* \text{ universally Radon-measurable on } S\}.$$

The characterization of  $\Lambda^{\text{URM}}(S)$  is not as straightforward as that of  $\Lambda^{\text{CBP}}(S)$ , even for the group  $\mathbb{Z}$  of integers with the discrete topology. On one hand, Glasner's proof of Theorem 2.1 in [10] demonstrates that if  $G$  is a countable discrete group then  $\Lambda^{\text{URM}}(\beta G) = G$ , which improves (for such groups) Corollary 3.3. On the other hand, the statement  $\Lambda^{\text{URM}}(\text{LUC}(\mathbb{Z})^*) = \mathbf{M}_t(\mathbb{Z})$ , which may be written simply as  $\Lambda^{\text{URM}}(\ell_\infty^*) = \ell_1$ , is neither provable nor disprovable in the ZFC set theory. That follows from old results about *medial limits*, covered by Fremlin [8, 538Q], along with a recent result of Larson [11].

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